

## Week 1: Algebra-Solutions

**Instructions:** The following consists of the solutions to the problems of the first week's algebra handout. For problems with official sources, a link to a solution is provided but for others, the complete solution is written. The problems covered here are meant to be of the same level as the NMTC descriptive questions and above that. Problems marked at the end with (\*) are a bit more tough (and fun!).

The problems covered here are meant to build your problem-solving skills, and as such, they are open-book. You are encouraged to use any textbook, video, or other online or offline resource as long as it doesn't give away the complete solution.

It is recommended that you try to attempt all the problems before reading the solutions.

## 1. Algebraic Manipulation

**Problem 1.1** Find all triples of real numbers  $a, b, c$  such that:

$$\begin{aligned}a + b &= a^2 + c^2 \\b + c &= b^2 + a^2 \\c + a &= c^2 + b^2\end{aligned}$$

**Solution.** Subtracting the first two equations gives us:

$$\begin{aligned}c - a &= a^2 - c^2 \\ \iff c &= a + a^2 - c^2\end{aligned}$$

Substituting this into the third equation gives us  $c = b^2$ . Similarly, we can get  $b = a^2$ ,  $a = c^2$ . Hence,  $c = c^6$ .

This gives us  $c = 0, 1$  which readily gives the only solutions as  $(a, b, c) = (0, 0, 0), (1, 1, 1)$

**Problem 1.2** (2025 BMO1 Q2) Find all real numbers  $x, y$ , and  $z$  such that:

$$\begin{aligned}x^2 + 2yz &= 4 \\y^2 + 2zx &= 4 \\z^2 + 2xy &= 1\end{aligned}$$

**Solution.** Official Video Solution.

**Problem 1.3** (2026 Kazakhstan) Let  $a$  and  $b$  be distinct real numbers such that

$$\frac{a}{b} + \frac{a + 10b}{b + 10a} = 2$$

Find  $\frac{a}{b}$

**Solution.** Rewrite the expression as

$$\frac{a}{b} + \frac{\frac{a}{b} + 10}{1 + 10\frac{a}{b}} = 2$$

Solving this quadratic in  $\frac{a}{b}$  gives us that  $\frac{a}{b} = 1, \frac{4}{5}$  but  $a$ , and  $b$  are distinct.

Hence  $\frac{a}{b} = \frac{4}{5}$  only.

**Problem 1.4** Let  $a, b, c$  be distinct nonzero real numbers such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

Prove that  $|abc| = 1$

**Solution.** We have:

$$\begin{aligned} a + \frac{1}{b} &= b + \frac{1}{c} \\ \iff bc &= \frac{b-c}{a-b} \end{aligned}$$

Similarly for the other 2 sets of equations. Multiplying all these equations together gives us that

$$(abc)^2 = \frac{b-c}{a-b} \cdot \frac{c-a}{b-c} \cdot \frac{b-a}{a-c} = 1$$

From which we get the desired result.

**Problem 1.5** Find all real numbers  $x, y, w, z$  such that:

$$\begin{aligned} x + y + z &= w \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{w} \end{aligned}$$

**Solution.** We have:

$$\begin{aligned}\frac{1}{x+y+z} &= \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \\ \iff xyz &= (xy + yz + zx)(x + y + z) \\ \iff 0 &= (x+y)(y+z)(z+x)\end{aligned}$$

WLOG, say  $y = -x$ . Then,  $z = w$  can be any real number. All in all, we have that:  $(x, y, z) = (a, -a, b)$  (And any permutations thereof) and  $w = b$  for any nonzero real numbers  $a, b$ .

**Problem 1.6** (OTIS) Factorise the polynomial:

$$a(b-c)^3 + b(c-a)^3 + c(a-b)^3$$

**Solution.** Seeing the cyclically symmetric<sup>1</sup> expression, it is natural to substitute  $a = b$  into the above expressive. Hence, we find that  $a - b$  is a factor. Similarly, so is  $b - c$  and  $c - a$ . Since our polynomial is cyclically symmetric and of degree 4, our last factor must be a cyclically symmetric polynomial of degree 1 multiplied by some constant, say  $\lambda$ . Hence, our polynomial factories to the expression:

$$\lambda(a-b)(b-c)(c-a)(a+b+c)$$

Plugging in  $x = 1, y = 2, z = 3$  allows us to deduce  $\lambda = 1$  and we're done.

**Problem 1.7** (CMIMC 2020 A7) Let  $x$  be a real number, solve the equation:

$$(x-1)(x-4)(x-2)(x-8)(x-5)(x-7) = -48\sqrt{3}$$

**Solution.** Official Solution.

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<sup>1</sup>A cyclically symmetric or cyclic polynomial is one where any cyclic permutation of the variables of the polynomial leave it unchanged.

**Problem 1.8** (Peru MO 2025 L3 P3) Determine all solutions  $(x, y, z)$  of the following system of equations:

$$\begin{aligned}\frac{y}{z} &= 3 - 2x \\ \frac{z}{x} &= 3 - 2y \\ \frac{x}{y} &= 3 - 2z\end{aligned}$$

assuming that  $x, y,$  and  $z$  are non-zero and  $xyz$  is an integer.

**Solution.** Observe that we have:

$$\begin{aligned}y &= 3z - 2xz \\ z &= 3x - 2yx \\ x &= 3y - 2zy\end{aligned}$$

Adding these equations together gives us  $x + y + z = xy + yz + zx$  (1). Further, multiplying the three original equations and simplifying while using (1) gives us:

$$x + y + z = \frac{13 - 4xyz}{3} \quad (2)$$

Even further beyond, observe that we have:

$$\begin{aligned}y^2 &= 3yz - 2xyz \\ z^2 &= 3xz - 2xyz \\ x^2 &= 3yx - 2xyz\end{aligned}$$

Adding these equations together gives us:

$$(x + y + z)^2 - 5(x + y + z) + 6xyz = 0$$

Substituting in the value of  $x + y + z$  and solving the quadratic in terms of  $xyz$  gives us  $xyz = 1$  (or  $xyz = -\frac{13}{8}$  which isn't an integer).

Hence, from (2) we deduce that  $x + y + z = 3$ . Now, note that:

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx) = 3^2 - 6 = 3 = xy + yz + zx$$

It follows that  $x = y = z = 1$ .

**Problem 1.9** (2010 AIME P9) Find the maximum possible value of  $a^3 + b^3 + c^3$  for real numbers  $a, b, c$  satisfying:

$$\begin{aligned}a^3 &= abc + 2 \\b^3 &= abc + 6 \\c^3 &= abc + 20\end{aligned}$$

**Solution.** Adding all three equations gives us:

$$a^3 + b^3 + c^3 = 3abc + 28$$

Hence, it suffices to find the maximum value of  $abc$ . Now note that multiplying all three equations gives a quadratic in terms of  $abc$ , which we solve to obtain the maximum value  $abc = -\frac{15}{7}$ . Hence, the maximum value of  $a^3 + b^3 + c^3$  is

$$28 - \frac{45}{7} = \frac{151}{7}$$

**Problem 1.10** (2024 Pakistan TSTST P3) Find all ordered triples of real numbers  $(x, y, z)$  satisfying:

$$\begin{aligned}xyz &= 1 \\x^2z - 2xz + 2z + xy &= 2\end{aligned}$$

**Solution.** Observe that the second equation is equivalent to:

$$\begin{aligned}x^2z^2 - 2xz^2 + 2z^2 + xyz &= 2z \\ \iff x^2z^2 - 2xz^2 + z^2 + z^2 - 2z + 1 &= 0 \\ \iff (xz - z)^2 + (z - 1)^2 &= 0\end{aligned}$$

By the trivial inequality, it follows that  $x = z = 1$ . From which it follows that  $y = 1$  as well.

**Problem 1.11** (2017 Indonesia MO P4) Determine all pairs of distinct real numbers  $(x, y)$  such that both of the following are true:

$$\begin{aligned}x^{100} - y^{100} &= 2^{99}(x - y) \\x^{200} - y^{200} &= 2^{199}(x - y)\end{aligned}$$

**Solution.** Start by making the substitution  $a = \frac{x}{2}, b = \frac{y}{2}$ . We have:

$$\begin{aligned}a^{100} - b^{100} &= a - b \\a^{200} - b^{200} &= a - b\end{aligned}$$

Hence,

$$a^{100} - b^{100} = a^{200} - b^{200} \implies a^{100} + b^{100} = 1 \quad (1)$$

WLOG, we assume that  $a > b$  and that  $a, b \neq 0$ . We then have:

$$\begin{aligned}a^{100} - b^{100} &> 0 \\ \iff a^{100} &> b^{100} \\ \iff |a| &> |b|\end{aligned}$$

but if  $a < 0$  then  $|a| < |b|$ . Which is a contradiction. Hence,  $a > 0$  always holds.

Now, assume  $b < 0$ , we have:

$$\begin{aligned}a^{100} &> a^{100} - b^{100} > a - b > a \\ \iff a^{100} &> a \\ \iff a^{99} &> a\end{aligned}$$

Hence,  $a > 1 \implies a^{100} > 1$  but by (1), this is a contradiction.

Hence,  $a, b > 0$ . Now note that

$$a^{100} + b^{100} = 1 = \frac{a^{100} - b^{100}}{a - b} = a^{99} + a^{98}b + \dots + b^{98}a + a^{99} > a^{99} + b^{99}$$

but from (1), we have that  $a, b < 1$ . Hence,  $a^{99} > a^{100}$  and  $b^{99} > b^{100}$ . Adding these equations, we get:

$$a^{99} + b^{99} > b^{100} + a^{100}$$

Which is another contradiction. Hence  $b = 0$  must hold which forces  $a = 1$ . Hence,  $(x, y) = (2, 0), (0, 2)$  is the only solution.

## 2. Sequences and Sums ft. Induction

**Problem 2.1** Calculate the following sums:

(a)  $\sum_{k=1}^n k! \cdot k$  (1969 Canada)

(b)  $\sum_{k=1}^n \frac{k}{(k+1)!}$

(c)  $\sum_{k=1}^n \frac{k+1}{(k-1)! + k! + (k+1)!}$

(d)  $\sum_{k=0}^{\infty} \frac{2^{2^k}}{4^{2^k} - 1}$  (IMC 2015)

**Solution.**

(a) Observe that  $k! \cdot k = (k+1)! - k!$ . Hence:

$$\sum_{k=1}^n k! \cdot k = (n+1)! - 1$$

(b) Observe that  $\frac{k}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}$ . Hence:

$$\sum_{k=1}^n \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$$

(c) Observe that  $\frac{n}{(n-2)! + (n-1)! + (n)!} = \frac{1}{(n-1)!} - \frac{1}{n!}$ . Hence:

$$\sum_{k=1}^n \frac{k+1}{(k-1)! + k! + (k+1)!} = 1 - \frac{1}{(n-1)!}$$

(d) Observe that  $\frac{2^{2^n}}{4^{2^n} - 1} = \frac{1}{4^{2^{n-1}} - 1} - \frac{1}{4^{2^n} - 1}$ . Hence:

$$\sum_{k=0}^{\infty} \frac{2^{2^k}}{4^{2^k} - 1} = \frac{1}{4^{2^{-1}} - 1} = 1$$

**Problem 2.2** Evaluate

$$\sum_{k=0}^n \frac{1}{(n-k)!(n+k)!}$$

**Solution.** The above fraction looks like a binomial coefficient, what if we make it one? We write:

$$\frac{1}{(n-k)!(n+k)!} = \frac{1}{(2n)!} \cdot \frac{(2n)!}{(n-k)!(n+k)!} = \frac{1}{(2n)!} \cdot \binom{2n}{n-k}$$

Hence we can write our sum as:

$$\begin{aligned} \sum_{k=0}^n \frac{1}{(n-k)!(n+k)!} &= \frac{1}{(2n)!} \cdot \sum_{k=0}^n \binom{2n}{n-k} \\ &= \frac{1}{(2n)!} \cdot \left( \sum_{k=0}^{2n} \binom{2n}{n-k} + \binom{2n}{n} \right) \\ &= \frac{1}{(2n)!} \cdot \left( (1+1)^{2n} + \binom{2n}{n} \right) \\ &= \frac{2^{2n-1}}{(2n)!} + \frac{1}{2(n!)^2} \end{aligned}$$

**Problem 2.3** (Source on AoPs) Evaluate the following sum:

$$\sum_{k=0}^n kx^{n-k}$$

**Solution.** Given the sum  $x^{n-1} + 2x^{n-2} + 3x^{n-3} + \dots$ , we can write it as a sum of the following:

$$\begin{aligned}
x^{n-1} + x^{n-2} + \cdots + x + 1 &= \left( \frac{1 - x^n}{1 - x} \right) \\
x^{n-2} + x^{n-3} + \cdots + x + 1 &= \left( \frac{1 - x^{n-1}}{1 - x} \right) \\
&\vdots \\
1 &= \left( \frac{1 - x}{1 - x} \right)
\end{aligned}$$

Adding these sums, we get

$$\begin{aligned}
x^{n-1} + 2x^{n-2} + 3x^{n-3} + \cdots &= \frac{(1 - x^n) + (1 - x^{n-1}) + \cdots + (1 - x)}{x - 1} \\
&= \frac{n - (x + x^2 + \cdots + x^n)}{x - 1} \\
&= \frac{n - \frac{x(x^n - 1)}{x - 1}}{x - 1} \\
&= \frac{x^{n+1} - nx + n - x}{(x - 1)^2}
\end{aligned}$$

And we're done.

**Problem 2.4** (2018 SEAMO) Evaluate the following series:

$$\frac{1}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \cdots + \frac{n^2}{2^n} + \cdots$$

**Solution.** Let the above sum be denoted by  $S$ . Observe that:

$$S - \frac{S}{2} = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \frac{7}{2^4} + \cdots$$

Further note that:

$$\left(S - \frac{S}{2}\right) - \frac{S - \frac{S}{2}}{2} = \frac{1}{2} + \frac{2}{2^2} + \frac{2}{2^3} + \cdots = \frac{3}{2}$$

Hence,  $S = 6$

**Problem 2.5** The sequence  $(a_n)_{n \in \mathbb{Z}_0^+}$  is defined by  $a_0 = 0$ ,  $a_1 = 1$  and for any integers  $m, n \geq 0$ , by  $a_{m+n} + a_m = a_n + a_{2m}$ . Find the value of  $a_{1434}$

**Solution.** Putting  $n = 0$  gives us  $2a_m = a_{2m}$ . Substituting this back in, our expression simplifies to:  $a_{m+n} = a_m + a_n$ .

We can now use induction to show that  $a_n = n$  for any natural number  $n$ . Hence,  $a_{1434} = 1434$

**Problem 2.6** (2009 Croatia) The sequence  $(a_n)_{n \in \mathbb{N}}$  is defined by  $a_1 = 1$ ,  $a_2 = 3$ , and  $a_n = a_{n-1} + a_{n-2}$ , for  $n \geq 3$ . Prove that  $a_n < \left(\frac{7}{4}\right)^n$

**Solution.** We proceed with strong induction. The base case of  $n = 3, 4$  is clear. Since  $a_n = a_{n-1} + a_{n-2}$ , it suffices to prove that:

$$\left(\frac{7}{4}\right)^{n-1} + \left(\frac{7}{4}\right)^{n-2} < \left(\frac{7}{4}\right)^n$$

Which follows from some algebra.

**Problem 2.7** (Source on AoPs) The sequence  $(a_n)_{n \in \mathbb{Z}_0^+}$  is defined by  $a_0 = a_1 = 1$  and  $a_{n+2} = a_{n+1} - \frac{a_n^2}{3}$  for natural number  $n$ . Show that  $a_n > 0$  for all  $n \in \mathbb{N}$

**Solution.** Click the source for the solution.

**Problem 2.8** Show that the sequence  $(a_n)_{n \in \mathbb{N}}$ , where  $\mathbb{N}$  defined by  $0 < a_1 < \frac{1}{2}$  and  $a_{n+1} = 2a_n(1 - a_n)$  is strictly increasing. That is, show that  $a_n < a_{n+1}$  always holds for any natural number  $n$ .

Hint: You may need to use the inequality  $\left(\frac{x+y}{2}\right)^2 \geq xy$  for positive reals  $x$  and  $y$  with equality only if  $x = y$

**Solution.** In order to show  $a_n < a_{n+1}$ , for all natural  $n$  it suffices to show that  $0 < a_n$  and  $1 < \frac{a_{n+1}}{a_n}$ . This is equivalent to showing:

$$1 < \frac{a_{n+1}}{a_n} = 2(1 - a_n) \iff a_n < \frac{1}{2}$$

We show this by induction. Assume that  $0 < a_n < \frac{1}{2}$ , then  $a_{n+1} = 2a_n(1 - a_n) > 0$ .

Further, note that we have:

$$a_{n+1} = 2a_n(1 - a_n) \leq 2 \left( \frac{a_n + 1 - a_n}{2} \right)^2 = \frac{1}{2}$$

Equality only holds when  $a_n = 1 - a_n \iff a_n = \frac{1}{2}$  but  $a_n < \frac{1}{2}$  always. Hence, we have that  $a_{n+1} < \frac{1}{2}$  and  $0 < a_{n+1}$ , as needed.

**Problem 2.9** (2025 Pakistan TST P3)(\*) Let  $(a_n)_{n \in \mathbb{Z}_0^+}$  be an infinite sequence of real numbers satisfying  $\frac{a_{n-1} + a_{n+1}}{2} \geq a_n$  for all positive integers  $n$ . Show that, for all positive integers  $n$ ,

$$\frac{a_0 + a_{n+1}}{2} \geq \frac{a_0 + a_1 + \cdots + a_n}{n}$$

**Solution.** No solution for this one! Figure it out yourself hehe.

**Problem 2.10** (2010 BMOSL A2)(\*) Let the sequence  $(a_n)_{n \in \mathbb{N}}$ , where  $\mathbb{N}$  denote the set of natural numbers, is given with  $a_1 = 2$  and  $a_{n+1} = a_n^2 - a_n + 1$ . Find the minimum real number  $L$ , such that for every  $k \in \mathbb{N}$

$$\sum_{i=1}^k \frac{1}{a_i} < L$$

**Solution.** Solution can be found here.

**Problem 2.11** (2014 IMO P1)(\*) Let  $(a_n)_{n \in \mathbb{N}}$  be an infinite sequence of positive integers. Prove that there exists a unique integer  $n \geq 1$  such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

**Solution.** Evan Chen's solution can be found here.